Assignment 2

1. The atmospheric pressure at sea level is about 1 bar.
   a. Calculate the Earth atmosphere's scale height h in units of km. Assume the gravitational acceleration, mean molecular weight, and the mean air temperature close to the Earth surface are all constants, and the latter is the freezing point of water.
   b. Choose your favorite plotting software and plot atmosphere pressure (y-axis) as a function of height above sea level (x-axis). Please set the x-axis limits to be 0 and $3h$.
   c. Calculate the pressure at the summit of Mount Everest (8848 m) and Mount Jing (景山, 90 meter above sea level), in units of bar.

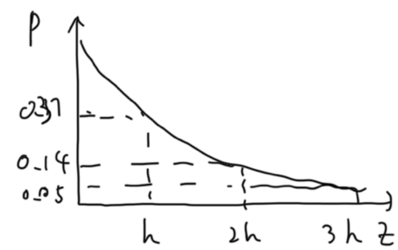
Solution:

a) From the lecture, we know the scale height in the Earth atmosphere

$$h = \frac{kT}{m_H g} = \frac{1.38\times10^{-16}\times 273}{29\times 1.67\times10^{-24}\times 9.8\times 100} \simeq 8\,\mathrm{km}$$

b) Pressure varies with height as

$p = p_0 e^{-\frac{z}{h}}$, $p_0 = 1$ bar at sea level



c) $p_{everest} = p_0 e^{-\frac{8848\,m}{h}} = 0.33$ bar

$p_{Jing} = p_0 e^{-\frac{90\,m}{h}} = 0.99$ bar.

2. 5G is the fifth and the latest generation wireless technology for digital cellular networks that began wide deployment in 2019. One frequency range of 5G network is from 24 GHz to 72 GHz. Calculate the corresponding wavelengths of this frequency range.

Solution: $\lambda = c/\nu$. Plug in the numbers, we get that 24 GHz corresponds to 1.25 cm, and 72 GHz corresponds to 0.42 cm.

3. Assume the Earth is a blackbody with a uniform surface temperature set by the balance between heating (receiving energy from the Sun) and cooling (losing energy from radiation).
   a. Calculate the equilibrium temperature of the Earth $T_\oplus$ (assume the Earth is on a circular orbit around the Sun). Express the results using only constants and $T_\odot$ (effective temperature of the Sun), $R_\odot$ (Solar radius), and a_0 (orbital radius of the Earth).

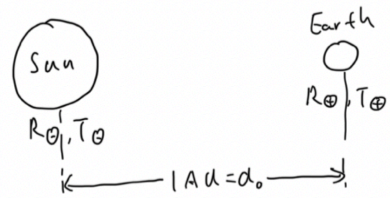
b. Plug in numbers and evaluate the equilibrium temperature of the Earth.

c. In reality, the Earth is not a blackbody. It has a Bond albedo of $A_b = 0.3$, meaning 30% of the solar radiation that hits the planet gets scattered back into space without been absorbed. Take this into account and reevaluate $T_\oplus$. (Assume the Earth still behaves as a blackbody at the frequency of its thermal emission)

Solution:

a) Luminosity of the Sun is

$$L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$$



Solar flux at a.

$$F_\odot = \frac{L_\odot}{4\pi a_0^2}$$

Heating of the Earth (energy received per second)

$$F_\odot \cdot \pi R_\oplus^2$$

Cooling of the Earth (energy lost per second)

$$\sigma T_\oplus^4 \, 4\pi R_\oplus^2$$

Heating balance cooling

$$F_\odot \pi R_\oplus^2 = \sigma T_\oplus^4 \, 4\pi R_\oplus^2$$

$$\Rightarrow \frac{4\pi R_\odot^2 \sigma T_\odot^4}{4\pi a_0^2} \pi R_\oplus^2 = \sigma T_\oplus^4 \, 4\pi R_\oplus^2$$

$$\Rightarrow T_\oplus = T_\odot \sqrt{\frac{R_\odot}{2 a_0}}$$

b) $T_\odot = 5777\ K$, $R_\odot = 676 \times 10^6\ m$, $a_0 = AU = 150 \times 10^9\ m$

$$\Rightarrow T_\oplus = 279\ K$$

c) In the above calculation, the heating term becomes

$$F_\odot \cdot 4\pi R_\oplus \cdot (1 - A_e)$$

Thus, $T_\oplus = T_\odot \sqrt{\frac{R_\odot}{2a_0}} \sqrt[4]{(1-0.3)} = 255\ K$

4.

a. Assume the Earth radiates at its effective temperature like a blackbody, based on the solution in problem 3c (assuming the effective temperature is about the same as the equilibrium temperature for the Earth), calculate the frequency, ν_{max}, and the corresponding wavelength $\lambda(\nu_{max})$, at which the brightness $B_\nu(T)$ peaks.
Hint: if you couldn't solve 3c, please look up the effective temperature of the Earth online.

b. Calculate λ_{max}, but for the brightness $B_\lambda(T)$ instead of $B_\nu(T)$.
c. If at $\nu = 1$ GHz, the Earth has an emissivity $\varepsilon_\nu = 0.1$. What is the brightness temperature of the Earth, T_b, at this frequency?
Hint 1: What is the definition of brightness temperature at a given frequency?
Hint 2: The Planck function has two well-known limits. Which limit applies in this case?

Solution:
(a) Equation (4.6) → $\nu_{max} = 1.5\times10^{13}$ Hz. This corresponds to $\lambda(\nu_{max}) = 20$ um.
(b) Equation (4.8) → $\lambda_{max} = 11$ um.
(c) The brightness temperature, T_b, is the temperature of a blackbody that has the same brightness at this particular frequency. The Earth's thermal emission peaks at 11 um, therefore at $\nu = 1$ GHz we are in the Rayleigh-Jeans tail ($\nu = 1$ GHz → $\lambda = 30$ mm >> λ_{max})

c) Earth's brightness at 1 GHz = $\underline{B_{\nu=1\text{GHz}}(T_e)} \cdot \varepsilon_\nu$

$kT_e \gg h\cdot 1\text{GHz}$, $B_\nu \rightarrow$ R-J. Law

$$= \frac{2\nu^2}{c^2} kT_e \cdot \varepsilon_\nu = \underline{\frac{2\nu^2}{c^2} kT_b}$$ definition of T_b

$\Rightarrow T_b = \varepsilon_\nu \cdot T_e = 25$ K

5. We have a cube with 20 cm on each side. The object behaves like a blackbody and has a temperature of 750 K.
a. Calculate the rate at which the cube emits energy into the space. Express the answer in units of W.
b. Given $B_\nu(T)$ as in Equation (4.3), derive $B_\lambda(T)$. Express the result using only constants, λ, and T.
c. How much energy does the cube emit per second at wavelengths between 4 um and 4.01 um? Express the answer in units of W.

Solution:

a) $E = \underbrace{A}_{\text{total area}} \cdot \underbrace{\sigma T^4}_{\text{power per unit area}} = 6 \times a^2 \cdot \sigma T^4$

$= 6 \times (20\,cm)^2 \times \sigma\,(750\,K)^4 = 4306\ W$

b) $B_\lambda(T) = B_\nu(T) \left|\frac{d\nu}{d\lambda}\right|$ (4.7)

$$\left|\frac{d\nu}{d\lambda}\right| = \left|\frac{d}{d\lambda}\left(\frac{c}{\lambda}\right)\right| = \frac{c}{\lambda^2}$$

$$B_\lambda(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \frac{c}{\lambda^2} \qquad \left(\nu = \frac{c}{\lambda}\right)$$

$$= \frac{2h}{c^2} \frac{c^3}{\lambda^3} \frac{c}{\lambda^2} \frac{1}{e^{\frac{h}{kT}\frac{c}{\lambda}} - 1} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{kT\lambda}} - 1}$$

A set of unit could be $W / sr / m^3$

c)

$\pi B_\lambda(T)$ (4.9) is the amount of energy emitted per unit surface area, per unit time, per unit wavelength. It is $B_\lambda(T)$ integrated over a hemisphere

Evaluate $\pi B_\lambda(T)$ at $\lambda = 4\,\mu m$ and $T = 750\,K$

$\pi B_\lambda(T) = 3045\ W/m^2 \cdot \mu m$

Therefore, the amount of energy emitted by the cube per second at $\lambda = 4 - 4.01\,\mu m$ is

$$P = \pi B_\lambda(T) \times \underbrace{A}_{\text{total surface area}} \times \underbrace{0.01\,\mu m}_{\Delta\lambda}$$

$$= 3045 \frac{W}{m^2\,\mu m} \times 6 \times (0.2\,m)^2 \times 0.01\,\mu m$$

$$= 7.3\ W$$

6. Calculate P_{out}/P_{in} for Saturn, where P_{out} is the amount of energy Saturn radiates to space per second, and P_{in} is the amount of energy Saturn receives from the Sun per second.

Hint: Appendix E may have some quantities that you could use.

Solution: there are two ways to calculate L_{out}/L_{in}.
This ratio is actually just $[T_{eff} / T_{eq}]^4$, where T_{eff} is the effective temperature and T_{eq} is the equilibrium temperature. Getting these two numbers from Table E.9, $L_{out}/L_{in} = [T_{eff} / T_{eq}]^4 = \mathbf{1.89}$.
Alternatively, $L_{out}/L_{in} = \sigma T_{eff}^4 \times 4\pi R_s^2 / ((1\text{-}A_b)L_\odot / 4\pi r^2 \times \pi R_s^2)$, where A_b is the Bond albedo, R_s is the radius of Saturn, r is the orbital radius of Saturn, and $L_\odot$ is the solar luminosity. Evaluating this equation gives $L_{out}/L_{in} = \mathbf{1.84}$.

7. Problem 6-12: Geysers have been observed on several bodies within the Solar System. The Prometheus plume on Io is 60 km high. Assuming that the height of the geyser is controlled only by ballistics, compute the exit velocity as the erupted material leaves the vent. Express your answer both as a velocity and as a ratio to Io's escape velocity.

Note: there are tables in Appendix E which may have information you need.

Solution: Equate the kinetic energy at the exit point with the potential energy at the height of the ballistic arc when $v = 0$: $mgy = 0.5mv^2$. Note that Io's radius is ~1800 km, much larger than the height of the plume, thus we can assume g is a constant. Using Table E.5., we have $g = GM/R_p^2 = 1.8$ m/s^2, thus $v = 464$ m/s. Escape velocity $v_e = (2GM/R_p)^{0.5} = 2558$ m/s. Thus $v/v_e = 0.18$.