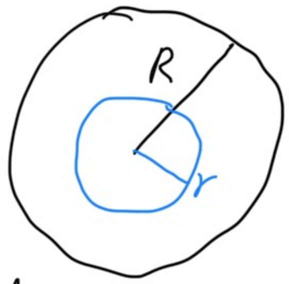
2026.03.16

Example 3

Estimate pressure at the center of a small planet

$$\frac{dP}{dr} = -\rho g \qquad \text{assume } \rho = \text{const}$$



$$g(r) = \frac{GM(r)}{r^2} = \frac{G}{r^2}\,\frac{4}{3}\pi\rho r^3 = \frac{4}{3}G\pi\rho r$$

$$\frac{dP}{dr} = -\rho\,\frac{4}{3}G\pi\rho r \Rightarrow \int_{P_c}^{0} dP = \int_0^R -\rho^2\,\frac{4}{3}G\pi r\,dr$$

$$\Rightarrow -P_c = -\frac{2}{3}\rho^2 G\pi R^2 \,, \qquad M = \frac{4}{3}\pi R^3\rho$$

$$P_c = \frac{3GM^2}{8\pi R^4}$$

Q: if now we shrink the size of the planet, how would P_c change?

A: P_c would go up.

Concentration towards the center → higher P_c .

Physics and Astrophysics (C3)

- Stellar Properties and Lifetimes (C.3.3)
  - virial theorem states that the time-averaged potential energy of a bound self-gravitating system is twice as large as the negative of the time-average of the kinetic energy of the system,

    $\langle E_G \rangle = -2 \langle E_K \rangle$ assignment 1, P6
  - Planet radius – gas giant planets have similar radii
    - Polytropy with n=1
    - Additional reading

    https://en.wikipedia.org/wiki/Lane–Emden_equation

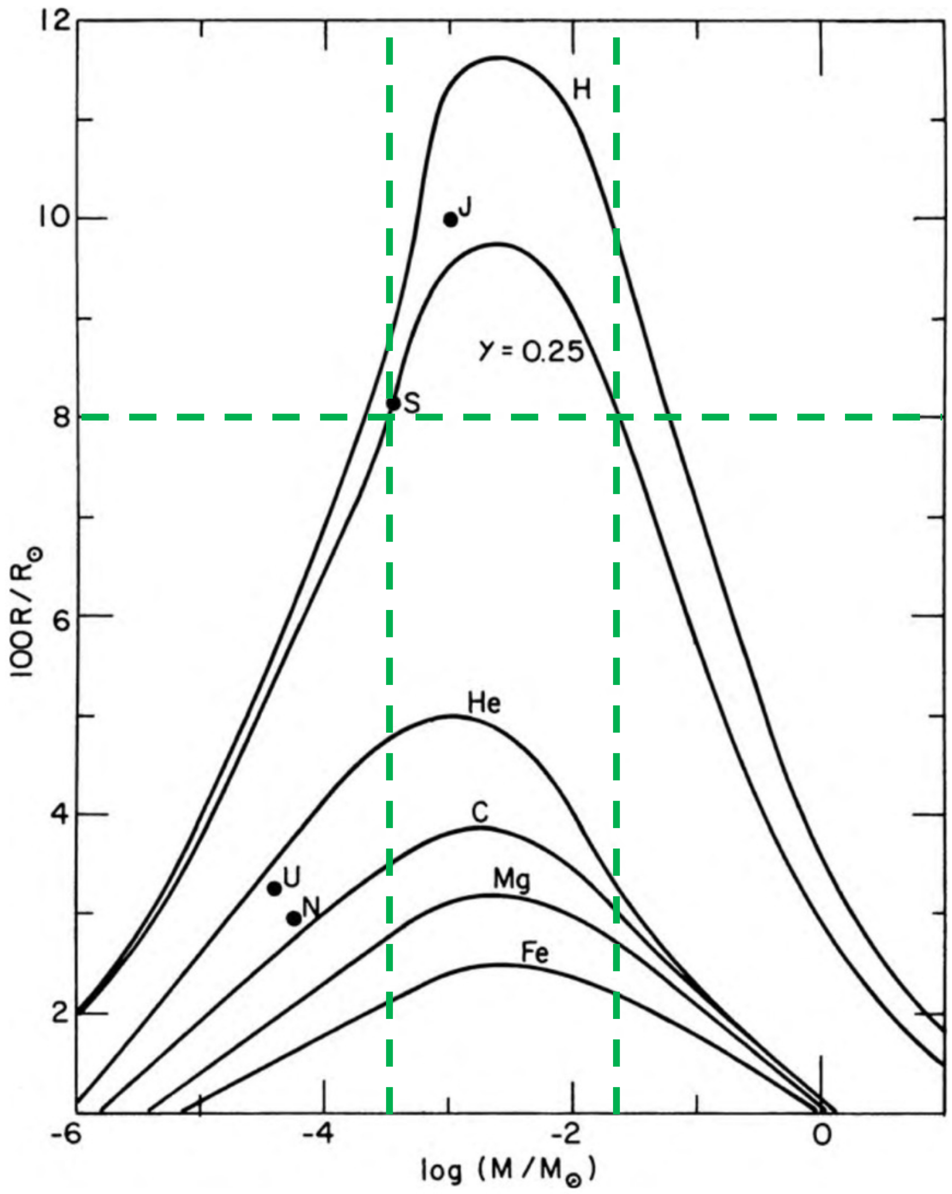
- Nucleosynthesis **(C.3.4)**

Gas giant planets have similar radii

Fig. 3.8

The mass–density relation for spheres of different materials at zero temperature, as calculated numerically using precise (empirical) equations of state. The *second curve* from the *top* is for a mixture of 75% H, 25% He by mass; all other curves are for planets composed entirely of one single element. The approximate locations of the giant planets are indicated on the graph. (Stevenson and Salpeter 1976)

Pressure increase, so does temperature. Eventually nuclear reaction begins.



Solar Heating and Energy Transport (C4)

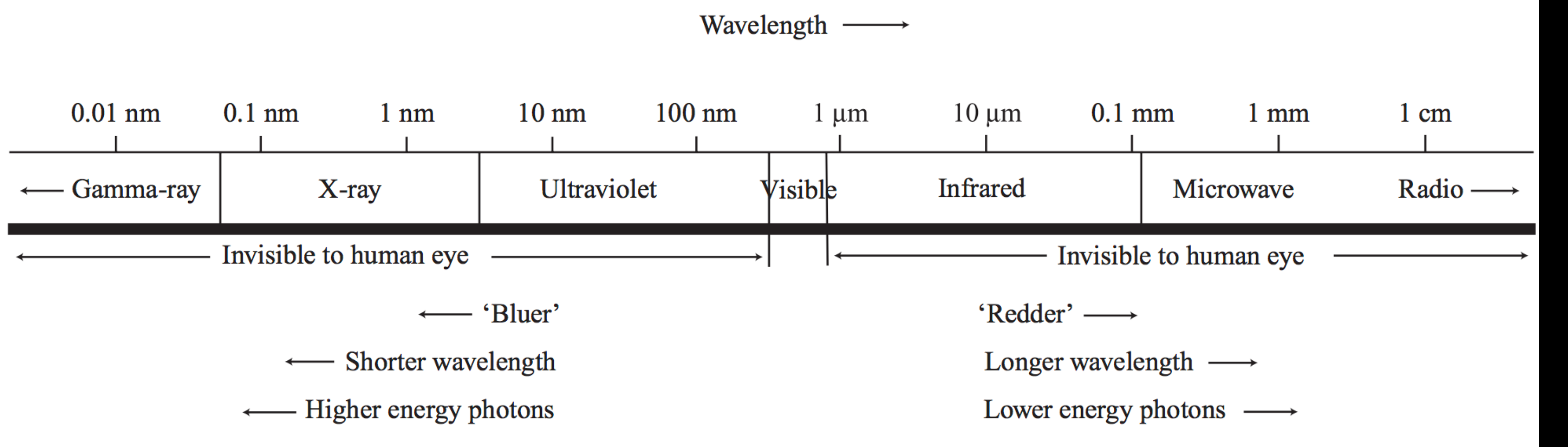


Figure 4.1 The electromagnetic spectrum. (Adapted from Hartmann 1989)

Supplementary reading

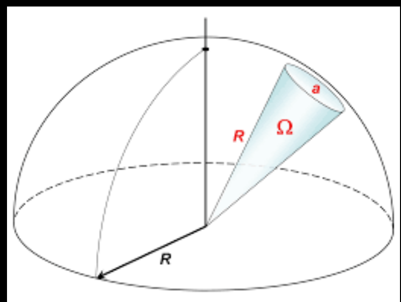
https://en.wikipedia.org/wiki/Kirchhoff%27s_law_of_thermal_radiation
https://en.wikipedia.org/wiki/Stefan–Boltzmann_law
https://www.tf.uni-kiel.de/matwis/amat/elmat_en/kap_3/basics/b3_2_2.html
https://en.wikipedia.org/wiki/Emissivity
https://en.wikipedia.org/wiki/Spectral_line
https://en.wikipedia.org/wiki/Limb_darkening
Convection instability https://websites.pmc.ucsc.edu/~glatz/astr_112/lectures/notes12.pdf

Blackbody Radiation (C.4.1.1)

- A blackbody is defined as an object that absorbs all radiation that falls on it at all frequencies and all angles of incidence; i.e., no radiation is reflected or scattered.
- $$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}$$

The power emitted by a given surface, per unit solid angle, per unit ***projected area***, per unit frequency.

- $B_\nu(T)$: brightness (J s^{-1} m^{-2} Hz^{-1} sr^{-1} or W m^{-2} Hz^{-1} sr^{-1})
- h: Planck constant
- v: frequency
- k: Boltzmann constant
- T: temperature



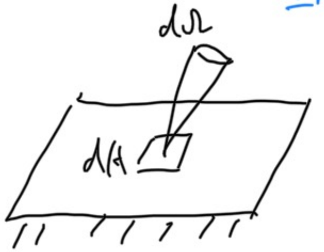
$$\Omega = \frac{a}{R^2}$$

Solid Angle

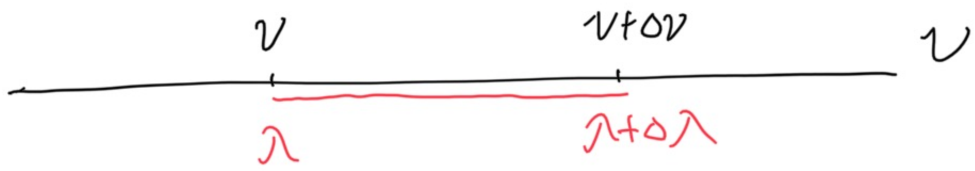
$$B_\nu(\nu,T): \frac{\text{Joule}}{s\ \ m^2\ \ Hz\ \ Sr} = \frac{dE}{dt\cdot dA\cdot d\nu\cdot d\Omega}, \quad J = \frac{kg\ m^2}{s^2}$$

$B_\nu(\nu,T)\cdot dt\cdot dA\cdot d\nu\cdot d\Omega$: energy emitted by dA at $t\to t+dt$ in $\nu\to\nu+d\nu$, in $d\Omega$ direction

In principle, $B = B(\nu, T, \Omega)$, although B happens to be independent of Ω



$$\Delta E = B_\nu\ \Delta\nu$$



$$\Delta E = B_\lambda \Delta\lambda$$

$B_\lambda(T)$

$$B_\nu\,\Delta\nu = B_\lambda\,\Delta\lambda \Rightarrow B_\lambda = \frac{\Delta\nu}{\Delta\lambda} B_\nu$$

$$\lambda\nu = c \Rightarrow d\lambda = -\frac{c}{\nu^2}d\nu$$

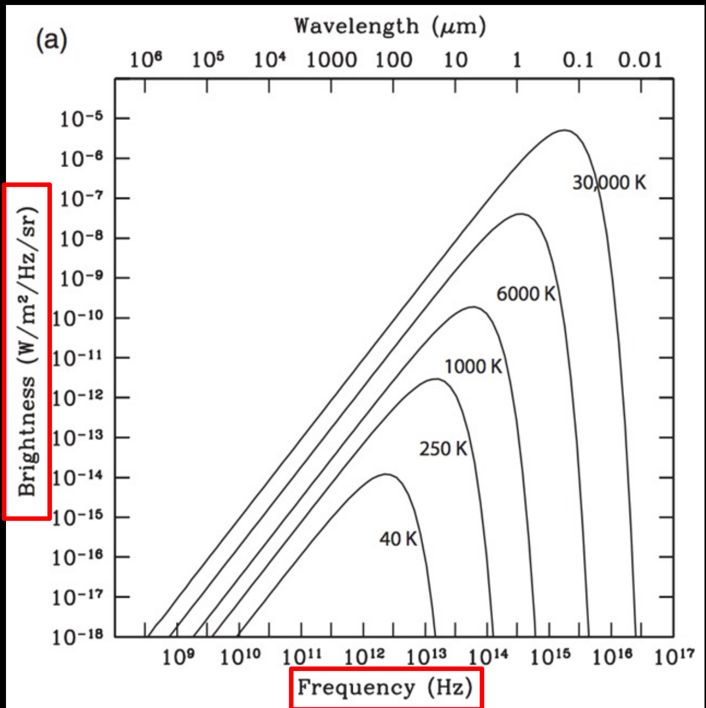
$$B_\lambda = \frac{\nu^2}{c} B_\nu \qquad (4.7)$$

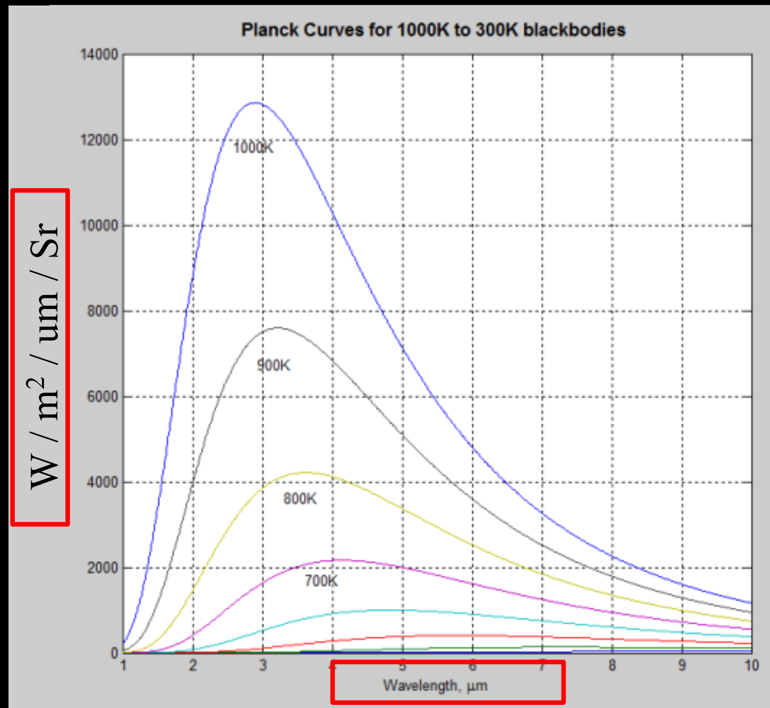
$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1},$$

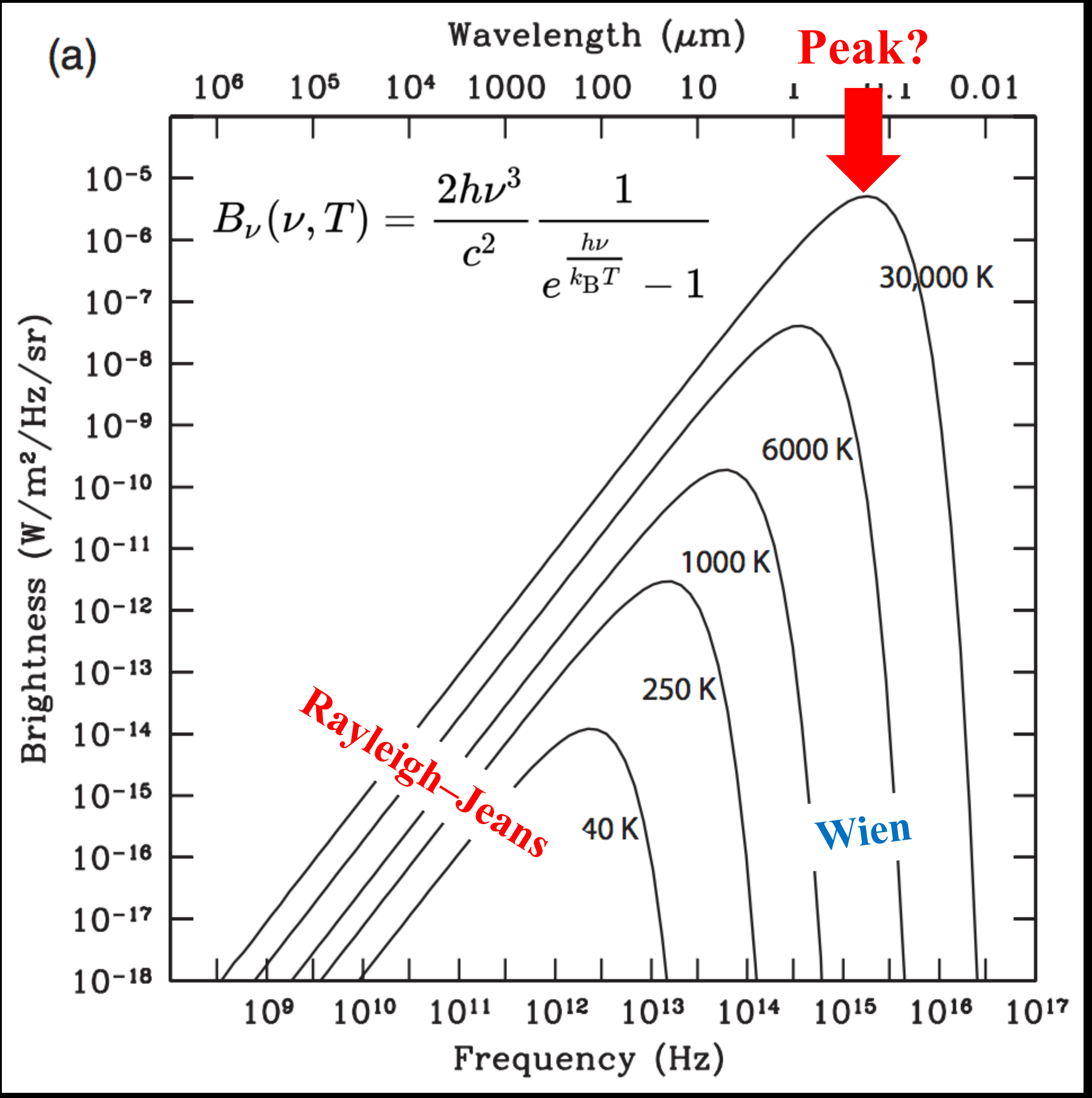
The power emitted per unit area, per unit solid angle, per unit **frequency**

The power emitted per unit area, per unit solid angle, per unit **wavelength**





https://scienceofdoom.com/2010/10/24/planck-stefan-boltzmann-kirchhoff-and-lte/



(a)
Wavelength (μm)
Peak?
10⁶ 10⁵ 10⁴ 1000 100 10 1 0.1 0.01
$B_\nu(\nu,T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}}-1}$
30,000 K
6000 K
1000 K
250 K
40 K
Rayleigh–Jeans
Wien
Brightness (W/m²/Hz/sr)
10⁻⁵ 10⁻⁶ 10⁻⁷ 10⁻⁸ 10⁻⁹ 10⁻¹⁰ 10⁻¹¹ 10⁻¹² 10⁻¹³ 10⁻¹⁴ 10⁻¹⁵ 10⁻¹⁶ 10⁻¹⁷ 10⁻¹⁸
10⁹ 10¹⁰ 10¹¹ 10¹² 10¹³ 10¹⁴ 10¹⁵ 10¹⁶ 10¹⁷
Frequency (Hz)

Limits

① Rayleigh-Jeans: $\underline{h\nu} \ll \underline{kT}$ characteristic E of a particle

energy of a photon

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1} \approx \frac{2h\nu^3}{c^2}\frac{1}{1+\frac{h\nu}{kT}-1} = \frac{2h\nu^3}{c^2}\frac{kT}{h\nu} = \frac{2\nu^2}{c^2}kT$$

$$e^{\ll 1} \approx 1+\frac{h\nu}{kT}$$

② Wien's law: $\underline{h\nu} \gg \underline{kT}$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1} \approx \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}} = \frac{2h\nu^3}{c^2}e^{-\frac{h\nu}{kT}}$$

$$e^{\gg 1} \gg 1$$

③ Peak in B_ν: $\frac{\partial B_\nu}{\partial \nu} = 0 \Rightarrow$ $\nu_{max} = 5.88\times10^{10}\,\frac{T}{K}$ (Hz)

$$h\nu_{max} = \boxed{2.8\,kT}$$

Peak in B_λ: $\frac{\partial B_\lambda}{\partial \lambda} = 0 \Rightarrow$ $\lambda_{max} = \frac{2.9\times10^{-3}}{T/K}$ (m)

$$h\nu(\lambda_{max}) = \boxed{5.0\,kT}$$

different

Check out https://en.wikipedia.org/wiki/Planck%27s_law#Peaks

C.4.1

- Stefan–Boltzmann law $F = \sigma T^4$: the total energy radiated per unit surface area of a black body per unit time across all wavelengths and all solid angle.

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Solid angle

Article Talk

From Wikipedia, the free encyclopedia

Not to be confused with spherical angle.

In geometry, a **solid angle** (symbol: Ω) is a measure of the amount of the field of view from some particular point that a given object covers. That is, it is a measure of how large the object appears to an observer looking from that point. The point from which the object is viewed is called the *apex* of the solid angle, and the object is said to *subtend* its solid angle at that point.

In the International System of Units (SI), a solid angle is expressed in a dimensionless unit called a *steradian* (symbol: sr). One steradian corresponds to one unit of area on the unit sphere surrounding the apex, so an object that blocks all rays from the apex would cover a number of steradians equal to the total surface area of the unit sphere, 4π. Solid angles can also be measured in squares of angular measures such as degrees, minutes, and seconds.

A small object nearby may subtend the same solid angle as a larger object farther away. smaller than the Sun, it is also much closer to Earth. Indeed, as viewed from any point o same solid angle (and therefore apparent size). This is evident during a solar eclipse.

Contents hide

In spherical coordinates there is a formula for the differential,

$$d\Omega = \sin\theta \, d\theta \, d\varphi,$$

where θ is the colatitude (angle from the North Pole) and φ is the longitude.

The solid angle for an arbitrary oriented surface S subtended at a point P is equal to the solid angle of the projection of the surface S to the unit sphere with center P, which can be calculated as the surface integral:

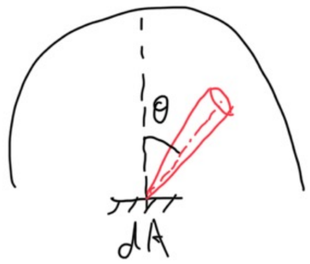
$$\Omega = \iint_S \frac{\hat{r} \cdot \hat{n}}{r^2} \, dS \; = \iint_S \sin\theta \, d\theta \, d\varphi,$$

Stefan–Boltzmann

Planck F $= E / t / A / \nu / \Omega$ (A: projected) units: $W/m^2/Hz/sr$

S-B law $= E / t / A$ (physical) W/m^2

To go from P.F. to S-B, integrate over ① all solid angle and ② frequency



① : Flux density

$$F_\nu = \frac{\int_{hemisphere} B_\nu(T)\,(dA\cos\theta)\,d\Omega}{dA}$$

$B_\nu(T)$: per projected area

$d\Omega = \sin\theta \cdot d\theta \cdot d\phi$ - - - - - wiki

$$= B_\nu(T) \int \cos\theta \sin\theta \, d\theta \, d\phi$$

independent of direction

$$\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta = \pi$$

$$= \pi B_\nu(T) \quad (4.10)$$

② $F = \int F_\nu \, d\nu = \pi \int_0^\infty B_\nu(\nu, T)\, d\nu$

F: S-B

$$= \pi \int \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu$$

integrable

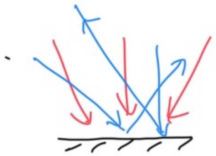
$$= \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 \Rightarrow F = \sigma T^4 \quad (4.11)$$

($\frac{2\pi^5 k^4}{15 c^2 h^3}$: σ)

C.4.1

- Stefan–Boltzmann law $F = \sigma T^4$: the total energy radiated per unit surface area of a black body per unit time across all wavelengths and all solid angle.
- Albedo (C.4.1.2)
  - Monochromatic albedo, A_ν: the ratio of the radiation reflected or scattered by the object to the total incident light at a given frequency ν
  - Bond albedo, A_b: the ratio of the total radiation reflected or scattered by the object to the total incident light (from the Sun)
  - Phase angle φ: the angle between the incoming and outgoing rays
  - Geometric albedo: $A_0(\varphi=0)$

Light at ν is incedent on a surface



$$A_\nu = \frac{\uparrow}{\uparrow + \uparrow}\Bigg|_{\nu} = \text{monochromatic albedo}$$

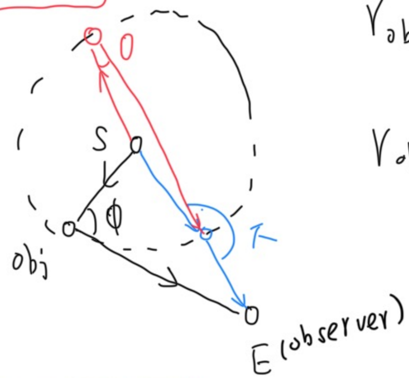
→ incident, absorbed

→ incident, scattered + reflected

$$\text{Bond } A_b = \frac{\uparrow}{\uparrow + \uparrow}\Bigg|_{\text{all } \nu} = \frac{\int_\nu \uparrow d\nu}{\int_\nu (\uparrow + \uparrow) d\nu} = \frac{\int_\nu A_\nu I_\nu \, d\nu}{\int_\nu I_\nu \, d\nu} = \text{Bond albedo}$$

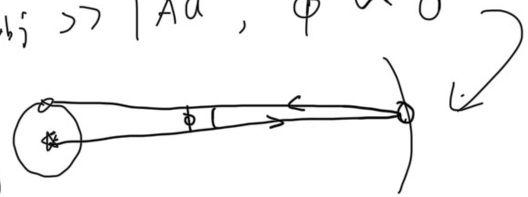
I_ν: determined by incident light, usually the Sun

phase angle



$r_{obj} < 1$ AU, $\phi \in [0, \pi]$

$r_{obj} >> 1$ AU, $\phi \sim 0$



geometric albedo

$$A_o = A_b(\phi = 0) = \frac{\uparrow}{\uparrow + \uparrow}\Bigg|_{\phi = 0}$$

Everything else being equal, the higher Bond albedo is, the cooler or warmer the object is?

https://en.wikipedia.org/wiki/Bond_albedo

Name	Bond albedo
Mercury [2] [3]	0.088
Venus [4] [3]	0.76
Earth [5] [3]	0.306
Moon[6] [6]	0.11
Mars [7] [3]	0.25
Jupiter [8] [3]	0.503
Saturn [9] [3]	0.342
Enceladus[10]	0.8
Uranus [11] [3]	0.300
Neptune [12] [3]	0.290
Pluto	0.4
Eris	0.96

C.4.1

- Stefan–Boltzmann law $F = \sigma T^4$: the total energy radiated per unit surface area of a black body across all wavelengths and all solid angle per unit time.
- Albedo (C.4.1.2)
  - Monochromatic albedo, A_ν: the ratio of the radiation reflected or scattered by the object to the total incident light at a given frequency ν
  - Bond albedo, A_b: the ratio of the total radiation reflected or scattered by the object to the total incident light (from the Sun)
  - Phase angle φ: the angle between the incoming and outgoing rays
  - Geometric albedo: $A_0(\varphi=0)$
- Temperature (C.4.1.3)
  - brightness temperature T_b: the temperature of a blackbody that has the same brightness at a particular frequency ν as the subject
  - effective temperature T_e: the temperature of a blackbody that emits the same total amount of electromagnetic radiation as the subject
  - equilibrium temperature T_{eq}: at the equilibrium temperature, the incoming solar radiation (insolation), F_{in}, is balanced, on average, by reradiation outwards, F_{out}

Brightness T

Brightness of an object is I_ν ($W/m^2/Hz/Sr$)

$dE/dt/dA/d\nu/d\Omega$

Solve $I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_b} - 1} \Rightarrow T_b$

If the obj is BB, then $T = T_b$. If not, $T \neq T_b$

T_b is a function of ν, $T_b(\nu)$

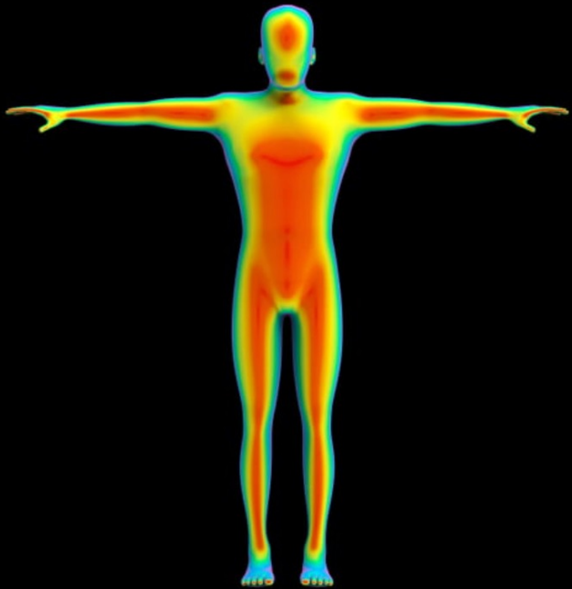
Effective T

Similar to T_B, But for flux $F = \int I_\nu \cdot d\nu \, d\Omega$ ($dE/dt/dA$, or W/m^2) instead of Brightness

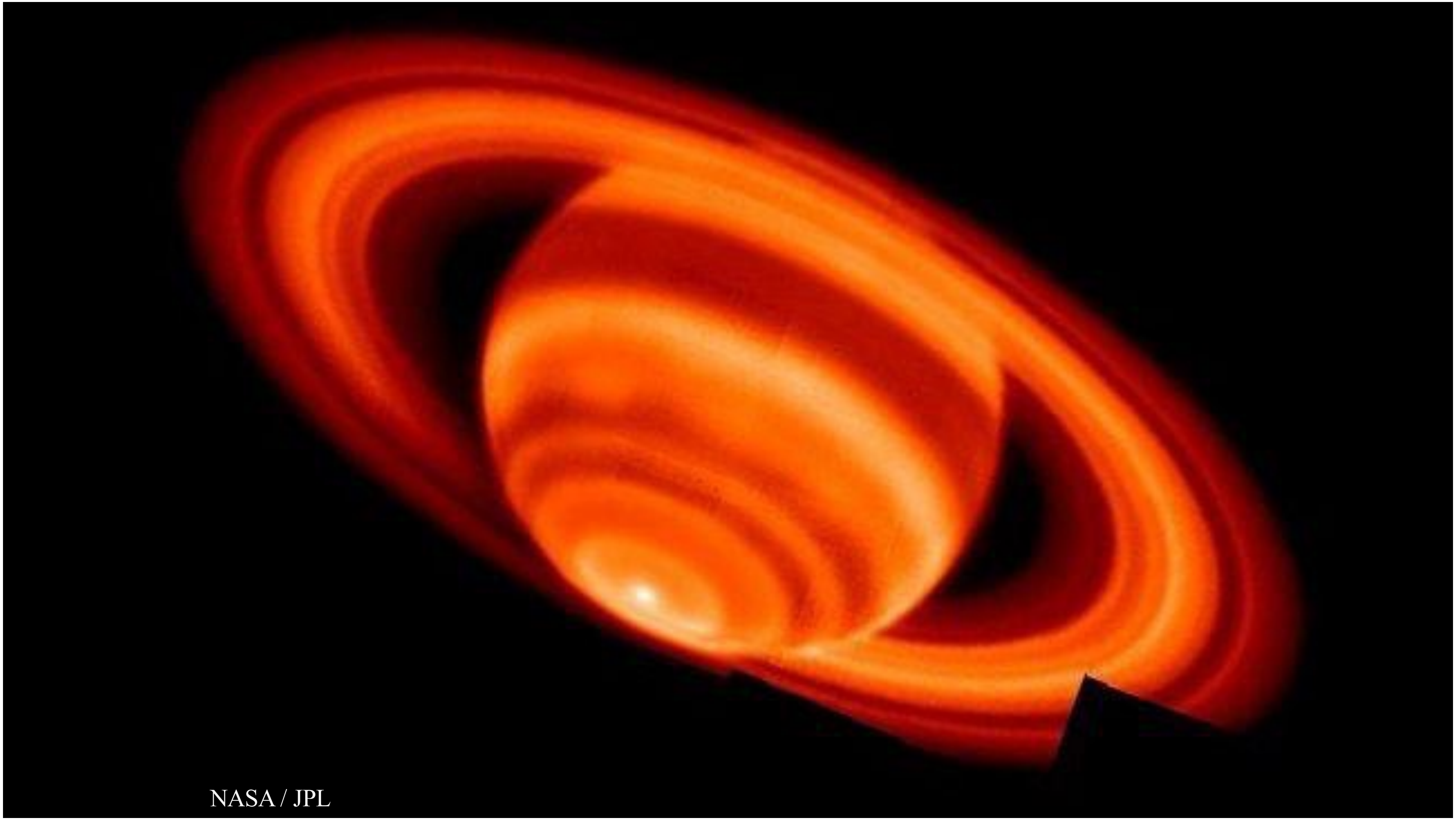
BB: $F_{BB}(T_e) = \sigma T_e^4 = F \Rightarrow T_e = \left(\frac{F}{\sigma}\right)^{1/4}$ (4.14)

For non BB $F = F_{BB} \cdot \varepsilon = \varepsilon \cdot (\sigma T_{phy}^4) = \sigma T_e^4$

$\Rightarrow T_{phy} = T_e \, \varepsilon^{-\frac{1}{4}}$



https://www.scienceabc.com/innovation/how-does-a-temperature-gun-infrared-thermometer-work.html#how-a-temperature-gun-works

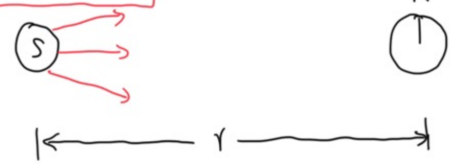

NASA / JPL

For a blackbody, albedo = 0 and emissivitiy = 1

For an object without an internal energy source, eventually its temperature reaches T_{eq}

Jupiter generates energy internally. It is still cooling down from its formation. It is also generating energy from contracting, by converting gravitational energy into heat.

Equilibrium T



Incoming energy received per s

$$P_{in} = (1-A_b)\frac{L_\odot}{4\pi r^2}\pi R^2$$

Outgoing $P_{out} = 4\pi R^2 \cdot \epsilon \cdot \sigma T^4$

Emissivity (≤ 1) | the ability to radiate may not be as good as BB

$\epsilon = \epsilon(\nu)$, similar to A_b, frequency averaged ϵ

At equilibrium, $P_{in} = P_{out}$

$$(1-A_b)\frac{L_\odot}{4\pi r^2}\pi R^2 = 4\pi R^2 \epsilon \sigma T^4$$

$$\Rightarrow \sigma T^4 = \frac{(1-A_b)L_\odot}{4\pi r^2 \cdot 4\epsilon}$$

$$\Rightarrow T = \left(\frac{(1-A_b)L_\odot}{16\pi r^2 \epsilon \sigma}\right)^{1/4} \equiv T_e$$

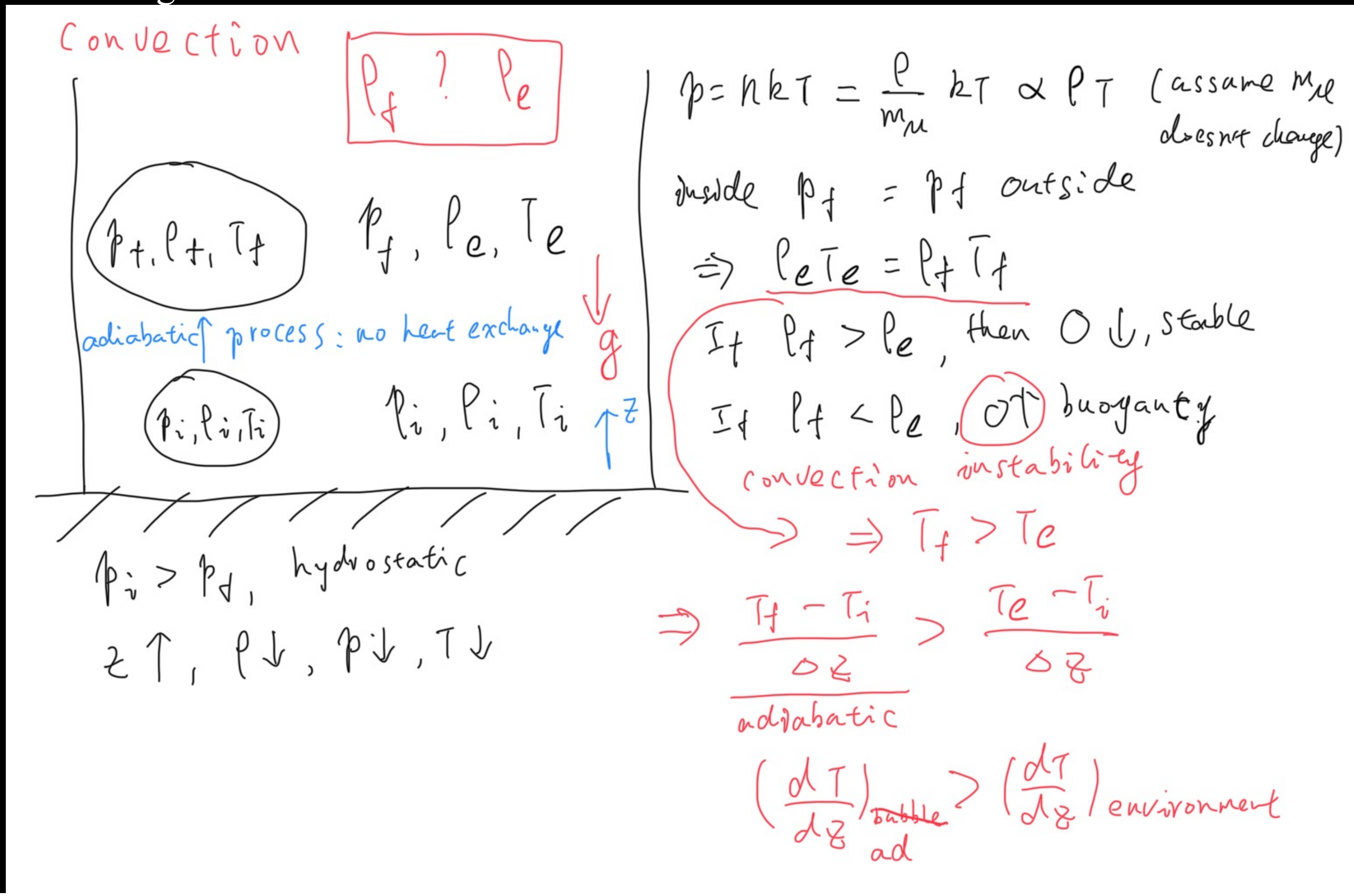
Earth: $P_{in} \simeq P_{out}$, $T \simeq T_e$, Income = Expense, healthy budget

Jupiter: $P_{in} < P_{out}$, $T > T_e$, Jupiter runs an energy deficit

- Three mechanisms to transport energy (C.4.2)
  - Conduction: the transfer of energy via collisions between molecules (C.4.3)
  - Convection: the transfer of energy due to the bulk movement of molecules within fluids (C.4.4)
  - Radiation : the transfer of energy via radiation (C.4.5)

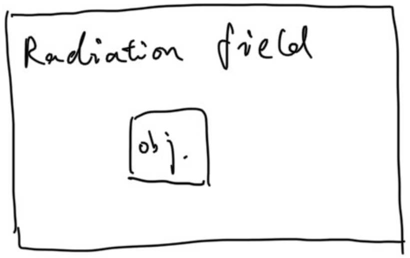
Convection is extremely efficient at transporting energy whenever the temperature gradient or lapse rate is superadiabatic

Convection thus effectively places an upper bound to the rate at which the temperature can decrease with height.



- Three mechanisms to transport energy (C.4.2)
  - Conduction: the transfer of energy via collisions between molecules (C.4.3)
  - Convection: the transfer of energy due to the bulk movement of molecules within fluids (C.4.4)
  - Radiation : the transfer of energy via radiation (C.4.5)
- Kirchhoff's law of thermal radiation: For an arbitrary body emitting and absorbing thermal radiation in thermodynamic equilibrium, the emissivity is equal to the absorptivity.
  - Thermodynamic equilibrium: in such a state there are no net macroscopic flows of matter or of energy, either within a system or between systems.
  - Emissivity: the ratio of the thermal radiation from a surface to the thermal radiation from an ideal black surface at the same temperature and same frequency
  - Absorptivity: ratio of the absorbed to the incident radiant power
  - Look up wiki for these concepts

Thermodynamic equilibrium



① within each obj. uniform T

if one part of an obj is hotter than the rest ✗

② no energy transport between obj.

if the obj gets energy from radiation field (being heated up) ✗

Basically, no macroscopic properties, e.g. T, change with time

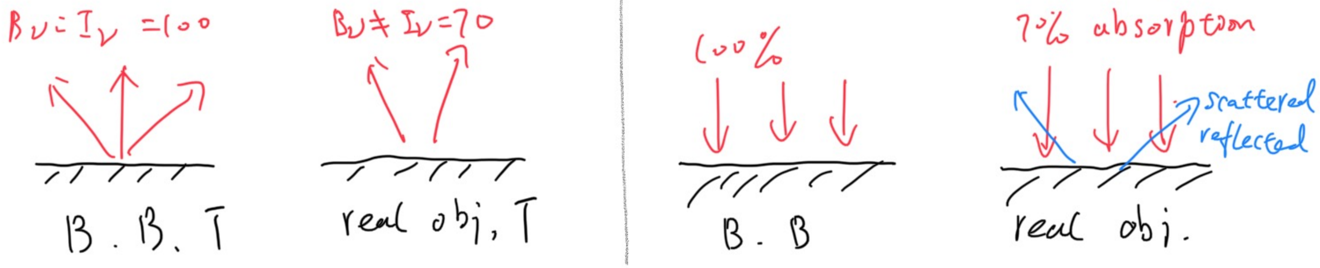
---

$\nu - \nu + d\nu$

emissivity$(T, \nu) = 70\%$

$\nu - \nu + d\nu$

Absorptivity $= 70\%$



K.C.: emissivity = absorptivity

C.4.5.1

Photons and Energy Levels in Atoms

Initial State

Emission

Final State

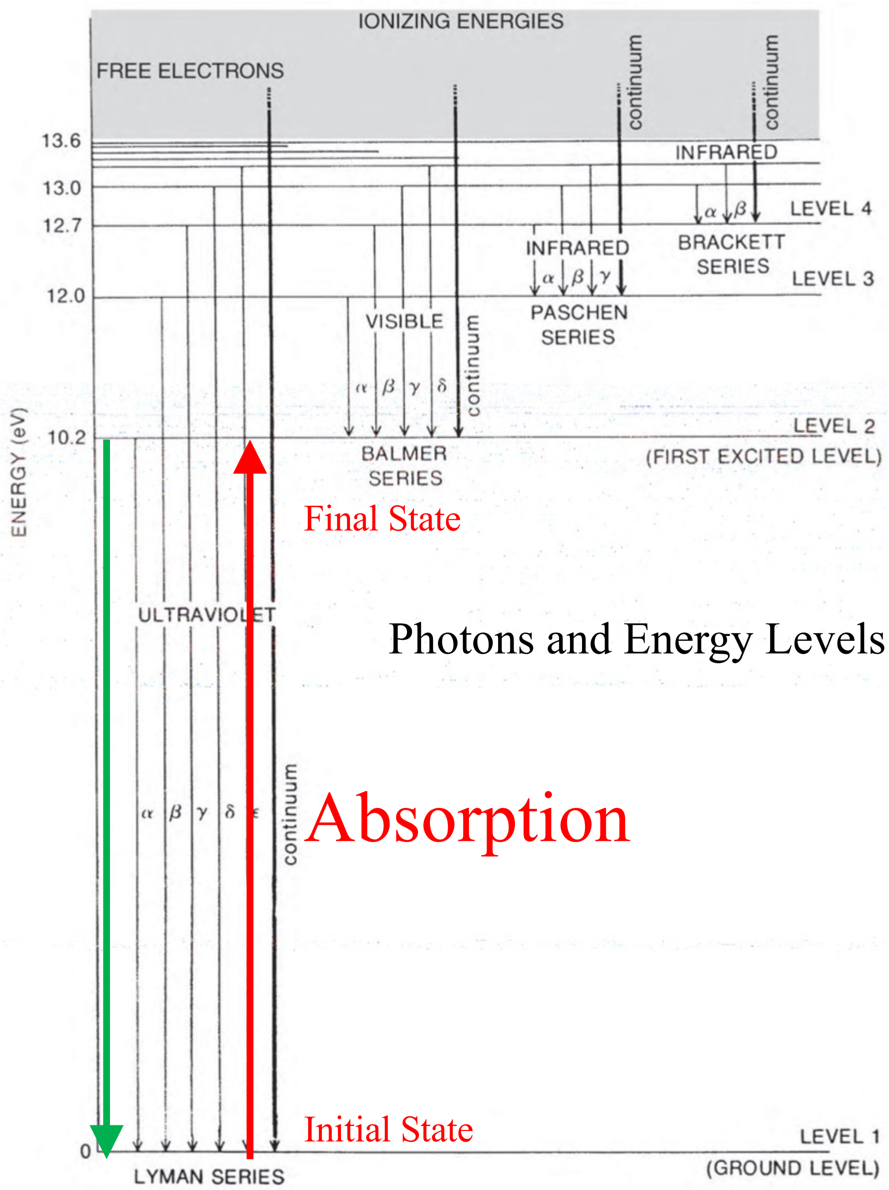
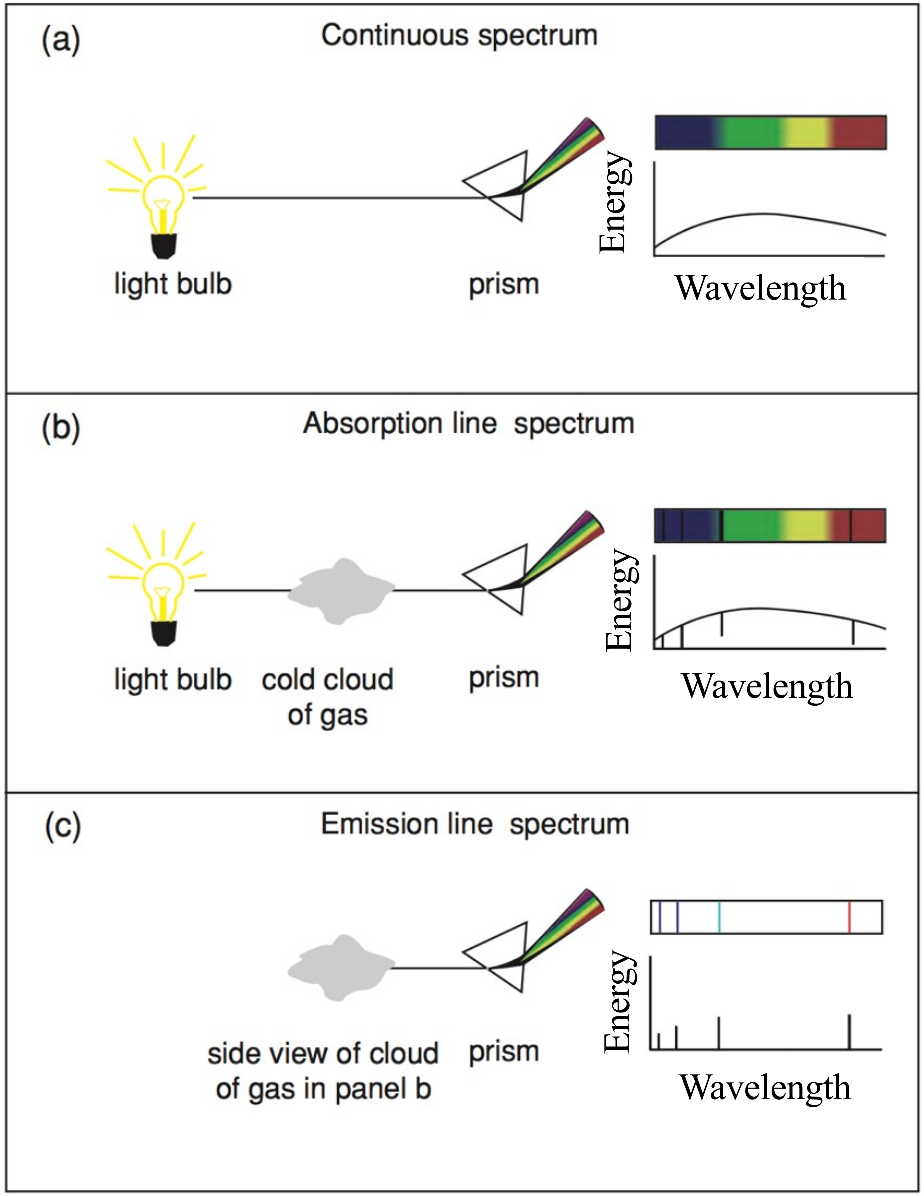


Figure 4.6 The energy levels of hydrogen and the series of transitions among the lowest of these energy levels. (Adapted from Pasachoff and Kutner 1978)



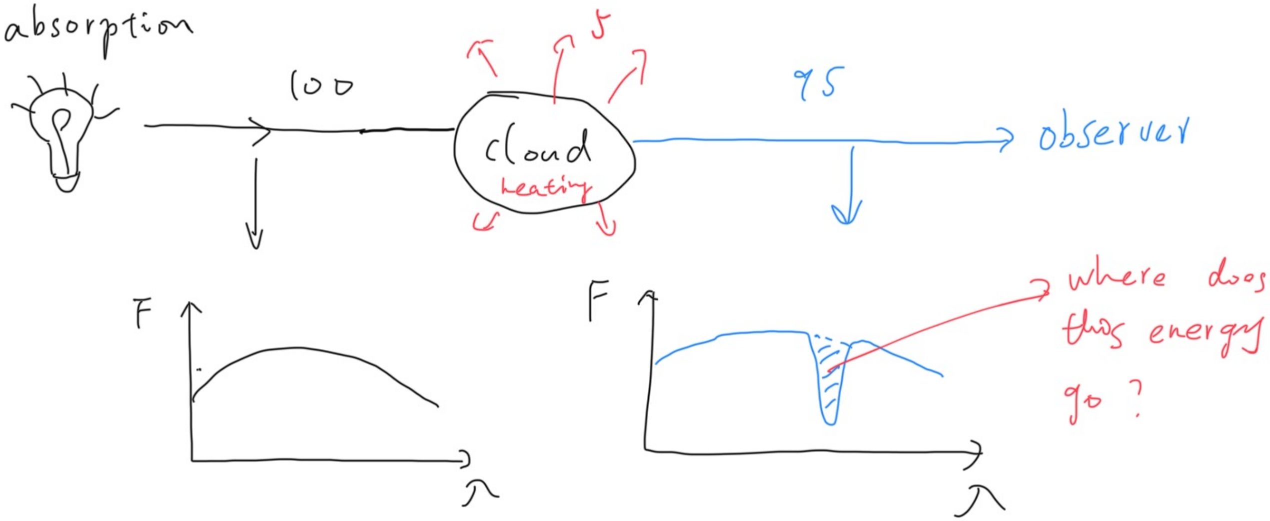
A solid, liquid or high-density gas produces (glows with) a continuous thermal spectrum.

A low-density cloud of gas between the observer and a hotter continuous spectrum source absorbs light at specific wavelengths.

A low-density cloud of gas produces emissions at specific wavelengths.

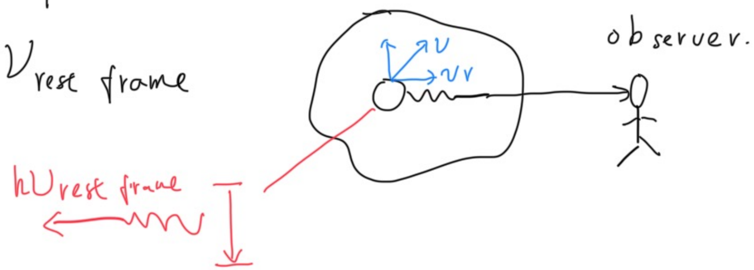
Fig. 4.7, C.4.5.2

absorption



Doppler shift

$$\Delta \nu = \frac{v_r}{c} \nu_{rest\ frame}$$



Greenhouse Effect (C.4.6)

Greenhouse Effect (C.4.6)