Assignment 1

Problem 1

Scale the entire solar system such that the diameter of the Earth is 1 cm.
1. Calculate the size of the Sun and the other planets.
2. Calculate the distances from the planets to the Sun.
3. How far is the nearest star to the Sun in this system?

Solution:

Solar System Object	Diameter	Distance	Model Object
Sun	1.1 m	–	very large beach ball
Mercury	3.7 mm	45 m	ball berring
Venus	9.5 mm	83 m	marble (slightly smaller)
Earth	10 mm	115 m	marble
Mars	5.2 mm	173 m	large ball berring
Asteroids	≤ 0.8 mm	330 ± 100 m	pinch of sand and dust
Jupiter	110 mm	600 m	softball
Saturn	92 mm	1.1 km	baseball, thin paper plate (rings)
Uranus	40.1 mm	2.2 km	ping-pong ball
Neptune	38.8 mm	3.5 km	20^{th} cent. ping-pong ball
Pluto, Eris	1.7 mm	4.5, 10 km	small beads
Other TNOs	≤ 1.2 mm	mostly 4 – 7 km	thimbleful of sand & dust
Proxima Cen.	160 mm	29,000 km	volleyball

Problem 2

Lagrangian points

a. L1, L2, and L3 are along the line joining two masses m_1 and m_2 in the circular restricted three body problem. Assuming R is the distance between the two main objects, and $m_2 << m_1$, find out the separation between L1 and m_2.
Hint 1: start from the definition of L points.
Hint 2: when $m_2 << m_1$, L1 and L2 are very close to m_2.

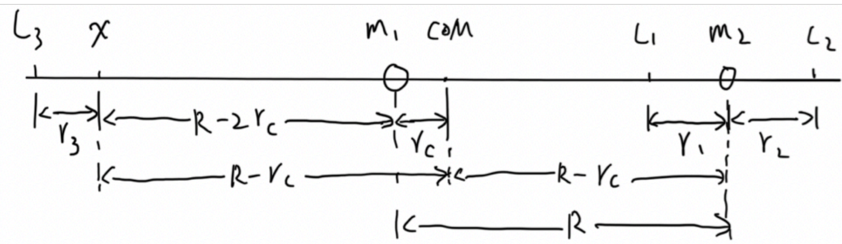
b. Find out the separation between L2 and m_2.

c. Evaluate L1 and L2 locations in the case of the Sun-Earth system. Express the results in units of both km and in Earth-Moon separation.

d. Is L3 closer or further away from the COM than m_2? Why?

e. L4 (and L5) forms an equilateral triangle with m_1 and m_2. Prove that it is a Lagrangian Point in the special case of $m_2 = m_1$.

Solution: In the schematic below, m_1 is the mass of the Sun, m_2 is the mass of the Earth, COM is center of mass. At all L points, an object rotates around the COM at the same angular velocity n as the Sun and the Earth, which is



$$n^2 = \frac{G(m_1+m_2)}{R^3}, \quad r_c = \frac{m_2 R}{m_1+m_2}$$

$\boxed{\text{At } L_1:}$

$$\frac{G(m_1+m_2)}{R^3}\underline{(R-r_1-r_c)} = \frac{Gm_1}{(R-r_1)^2} - \frac{Gm_2}{r_1^2}$$

$$R - r_1 - \frac{m_2 R}{m_1+m_2} = \frac{m_1 R}{m_1+m_2} - r_1$$

$$\frac{Gm_1}{R^2} - \frac{G(m_1+m_2)}{R^3} r_1 = \frac{Gm_1}{\underline{(R-r_1)^2}} - \frac{Gm_2}{r_1}$$

$$\because m_2 \ll m_1 \quad \therefore r_1 \ll R \Rightarrow \frac{1}{(R-r_1)^2} = \frac{1}{R^2}\left(1-\frac{r_1}{R}\right)^{-2} \approx \underline{\frac{1}{R^2}\left(1+\frac{2r_1}{R}\right)}$$

$$\frac{Gm_1}{R^2} - \frac{Gm_1 r_1}{R^3} - \frac{Gm_2}{R^3} r_1 = \frac{Gm_1}{R^2}\left(1+2\frac{r_1}{R}\right) - \frac{Gm_2}{r_1^2}$$

$$-\frac{3Gm_1 r_1}{R^3} - \frac{Gm_2}{R^3} r_1 = -\frac{Gm_2}{r_1^2}$$

$$r_1 = R\sqrt[3]{\frac{m_2}{3m_1+m_2}} \approx \boxed{R\sqrt[3]{\frac{m_2}{3m_1}}}$$

$\boxed{\text{At } L_2}$

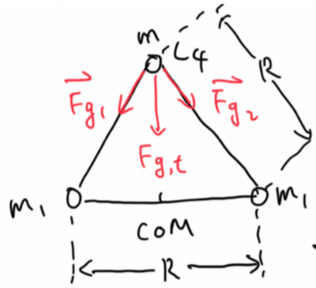
$$\frac{G(m_1+m_2)}{R^3}(R-r_c+r_2) = \frac{Gm_1}{(R+r_2)^2} + \frac{Gm_2}{r_2^2}$$

$$\text{similarly}, \Rightarrow r_2 \approx r_1 \approx \boxed{R\sqrt[3]{\frac{m_2}{3m_1}}}$$

c): L1 and L2 are both about 1.5 million kilometers away from the Earth, or about 4 times the Earth-moon separation.

d) Point $\boldsymbol{x}$ is at a symmetric location from the COM with respect to $\boldsymbol{m_2}$. An object placed at $\boldsymbol{x}$ is subject to a greater gravitational force (pointing to the COM) than $\boldsymbol{m_2}$, because (1) the separation between $\boldsymbol{x}$ and $\boldsymbol{m_1}$ is smaller than the distance between $\boldsymbol{m_2}$ and $\boldsymbol{m_1}$, thus $\boldsymbol{m_1}$ provides a greater gravitational force to $\boldsymbol{x}$ than to $\boldsymbol{m_2}$, and (2) $\boldsymbol{x}$ is subject to an additional gravitational force by $\boldsymbol{m_2}$. Therefore, an object placed at $\boldsymbol{x}$ must be rotating around the COM faster than $\boldsymbol{m_2}$. Thus, L3 must be further away from the COM than $\boldsymbol{x}$.

$\boxed{L_4, L_5}$



$$|\vec{F}_{g_1}| = |\vec{F}_{g_2}| = \frac{Gm_1 m}{R^2}$$

$$\vec{F}_{g_1} + \vec{F}_{g_2} = \sqrt{3}\,\frac{Gm_1 m}{R^2}$$

To orbit around the COM at the same angular velocity as the two m_1, $n^2 = \frac{2Gm_1}{R^3}$

Object m needs to be pull towards the COM by a force

$$F = n^2 \frac{\sqrt{3}}{2} R \cdot m = \frac{2Gm_1}{R^3}\frac{\sqrt{3}}{2} R\, m = \sqrt{3}\,\frac{Gm_1 m}{R^2}$$

Problem 3

What's the time interval between two consecutive tides induced by the moon?

Hint 1: how many tides induced by the moon between two successive moonrises?
Hint 2: You might want to think about time in sidereal time. If you are unfamiliar with the concept, Wikipedia and Figure 2.21 in the book might be helpful.

Solution: Sidereal time is a "time scale that is based on Earth's rate of rotation measured relative to the fixed stars",[1] or more correctly, relative to the March equinox.
A sidereal month $P_m = 27.321661$ days, and a sidereal day $P_d = 23.9344696$ hours. Successive moonrises are separated by $\Delta t = (P_d^{-1} - P_m^{-1})^{-1} = 24.84$ hours. Successive tides induced by the moon are separated by $\Delta t/2$=12.42 hours.

Problem 4

The sun is losing 6×10^{12} grams of mass every second at the moment via its solar wind and by converting mass into radiation. The Earth orbits the Sun. As the mass of the Sun decreases, the Earth is held a bit less strongly, and its orbit expands.

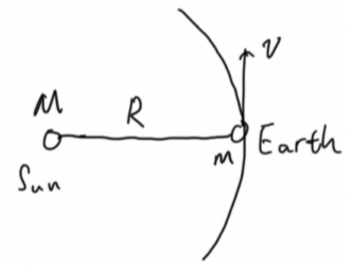
a. Derive Eqn. (2.65) in the textbook. You can assume circular orbits.
b. Evaluate the expansion rate in units of cm/yr.

Hint: the orbital angular momentum of the Earth is conserved in this process.

Solution:

Orbital velocity v satisfies

$$m\frac{v^2}{R} = \frac{GMm}{R^2} \Rightarrow v = \sqrt{\frac{GM}{R}}$$



Orbital angular momentum of the Earth h is

$$h = m\cdot v\cdot R = mR\sqrt{\frac{GM}{R}} = m\sqrt{GMR} \Rightarrow R = \frac{h^2}{m}\frac{1}{GM}$$

h is conserved, thus $\dfrac{dR}{dt} = -\dfrac{h^2}{Gm}\dfrac{dM}{dt}\dfrac{1}{M^2}$

or $$\frac{dR}{dt} = -\frac{R}{M}\frac{dM}{dt} = -\frac{1.5\times10^{13}\,\text{cm}}{2\times10^{33}\,\text{g}}\,\frac{6\times10^{12}\,\text{g}}{\text{s}} \approx 1.5\ \text{cm/yr.}$$

Problem 5

A small asteroid on a circular orbit around the Sun releases a dust grain. Assuming at the moment of release the dust grain achieves the same orbital velocity around the Sun as the asteroid, and the mass of the asteroid is too small to affect the dynamics of the grain. What's the minimum β value (equation 2.60) that the dust grain needs to have in order to escape the solar system? Ignore all other bodies in the Solar system (except the Sun of course). Please derive your answer instead of directly giving a number.

Solution: For a dust with β and mass m, it sees a Solar mass of (1- β) $M_\odot$ (right below Equation 2.61). The total energy of this dust at distance R from the Sun with the same velocity as the asteroid is

$$\begin{aligned} E_{tot} &= E_{potential} + E_{kinetic} \\ &= -\frac{GM_\odot(1-\beta)m}{R} + \frac{1}{2}mv^2, \quad v^2 = \frac{GM_\odot}{R} \\ &= -\frac{GM_\odot(1-\beta)m}{R} + \frac{1}{2}m\frac{GM_\odot}{R} \\ &= -\frac{GM_\odot(1-2\beta)m}{R} \end{aligned}$$

Now, for this dust particle to escape the Solar system, we need its total energy to be positive

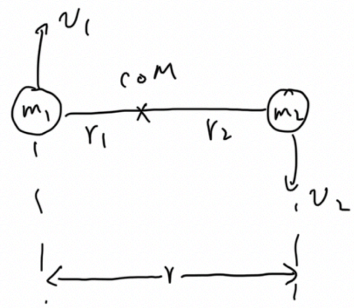
$$E_{tot} \geq 0 \Rightarrow 2\beta-1 \geq 0 \Rightarrow \beta \geq \frac{1}{2}$$

Problem 6

Prove the virial theorem in the case of m_1 and m_2 orbiting each other in circular orbits in the general case, i.e., without assuming $m_1 >> m_2$. The two objects are separated by a distance r.

1. What is the total kinetic energy of the system?
2. What is the total gravitational potential energy?
3. Show that the virial theorem holds.

Solution:



The two objects are doing circular orbits around their Center of Mass.

Force balance on m_1: $\frac{G m_1 m_2}{r^2} = m_1 \frac{v_1^2}{r_1} \Rightarrow v_1^2 = \frac{G m_2}{r^2} r_1$

Similarly $v_2^2 = \frac{G m_1}{r^2} r_2$

a) $E_k = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$$= \frac{1}{2} m_1 \frac{G m_2}{r^2} r_1 + \frac{1}{2} m_2 \frac{G m_1}{r^2} r_2$$

$$= \frac{1}{2} \frac{G m_1 m_2}{r^2} \underbrace{(r_1 + r_2)}_{=r} = \frac{1}{2} \frac{G m_1 m_2}{r}$$

b) $E_g = -\frac{G m_1 m_2}{r}$

c) $E_g = -2 E_k$, Virial theorm holds